

# DESIGN OF PARAMETERIZED STATE OBSERVERS AND CONTROLLERS FOR A CLASS OF NONLINEAR CONTINUOUS-TIME SYSTEMS

A. Alessandri, C. Cervellera, D. Macciò, and M. Sanguineti

**Abstract**—The design of observers and controllers for a class of continuous-time, nonlinear dynamic systems with Lipschitz nonlinearities is addressed. Observers and controllers that depend on a linear gain and a parameterized function implemented by a feedforward neural network are considered. The gain and the weights of the neural network are optimized in such way to ensure the convergence of the estimation error for the observer and the stability of the closed-loop system for the controller, respectively. This is achieved by constraining the derivative of a quadratic Lyapunov function to be negative definite on a grid of points, penalizing the constraints that are not satisfied. It is shown that suitable sampling techniques such as low-discrepancy sequences, commonly employed in quasi-Monte Carlo methods for high-dimensional integration, allow one to reduce the computational burden required to optimize the network parameters. Simulations results are presented to illustrate the effectiveness of the method.

## I. INTRODUCTION

Observers and controllers for nonlinear systems can be designed on the basis of a Lyapunov function that guarantees the stability of the estimation error or of the state, respectively. Unfortunately, there is no general methodology to find such Lyapunov function.

First results on observers for nonlinear systems were presented in [1], [2]. Later on, the state-space transformation approach was proposed [3], [4]; more recently, high-gain [5] and variable-structure [6] observers were presented. As to control, a critical point is that of proving stability of feedback law for general nonlinear systems, which is usually accomplished by finding suitable Lyapunov functions. Among the various approaches, we shall refer to [7], [8], where the use of parameterized functions allows to construct candidate Lyapunov functions, whose parameters are optimized to ensure the positive definiteness of such functions and the negative definiteness of their derivatives.

Taking the results of [9], [10] as departure points, both state estimation and control problems are separately consid-

This work was supported by the EU and the Regione Liguria through the Regional Programmes of Innovative Action (PRAI) of the European Regional Development Fund (ERDF) and by PRIN grant of the Italian Ministry for University and Research, Project “New Techniques for the Identification and Adaptive Control of Industrial Systems.”

A. Alessandri is with Department of Production Engineering, Thermoenergetics, and Mathematical Models (DIPTM), University of Genoa, P.le Kennedy Pad. D, 16129 Genoa, Italy [alessandri@diptem.unige.it](mailto:alessandri@diptem.unige.it)

C. Cervellera and D. Macciò are with Institute of Intelligent Systems for Automation (ISSIA-CNR), National Research Council of Italy, Via De Marini 6, 16149 Genoa, Italy [cervellera,ddmach@ge.issia.cnr.it](mailto:cervellera,ddmach@ge.issia.cnr.it)

M. Sanguineti is with Department of Communications, Computer and System Sciences (DIST), University of Genoa, Via Opera Pia 13, 16145 Genoa, Italy [marcello@dist.unige.it](mailto:marcello@dist.unige.it)

ered in this paper for continuous-time, nonlinear dynamic systems by means of a class of parameterized functions (e.g., feedforward neural networks) that allows to take on the structure of the observer and of the controller, respectively. The design parameters (i.e., the coefficients of the linear gain and the weights of the neural network) can be chosen in such a way to constrain the derivative of a quadratic Lyapunov function to be negative on a sampling grid of points on the involved spaces. This is accomplished by minimizing a cost function that penalizes the constraints that are not satisfied in correspondence to the sampling points. It is worth noting that the selection of the design parameters is made completely off line, which is the main advantage with respect to most neural approaches to estimation and control for nonlinear systems that rely on the on-line adaptation of the neural weights.

Under assumptions on the distribution of the sampling points and smoothness of the Lyapunov function, the negative definiteness of the derivative of the Lyapunov function is ensured. In particular, special deterministic sequences that aim at optimizing “how uniformly” the points are spread in the space are shown to deterministically provide convergence [11], [12], [13].

The work is organized as follows. Section II is devoted to the description of the estimation problem; the construction of the observer is discussed, together with the related convergence results. In Section III it is shown how the same methodology can be applied to the design of controllers. Section IV is a discussion on low-discrepancy sequences, i.e., the kind of deterministic sampling we employ to ensure the desired convergence properties. The class of approximating networks used to design the observer and the controller is described in Section V. Simulation results are shown in Section VI.

## II. DESIGN OF OBSERVERS WITH GUARANTEED STABILITY

### A. System description

We consider quite a general class of nonlinear continuous-time systems given by

$$\begin{cases} \dot{x} = Ax + f(x) \\ y = Cx \end{cases}, \quad t \geq 0, \quad (1)$$

where  $x(t) \in X \subset \mathbb{R}^n$  is the vector to be estimated,  $X$  is a connected, compact set, and  $y(t) \in \mathbb{R}^m$  is a vector of measures.  $A \in \mathbb{R}^{n \times n}$ ,  $C \in \mathbb{R}^{m \times n}$ , and  $f : X \rightarrow \mathbb{R}^n$  are supposed to be known. In order to guarantee

the existence and the uniqueness of the solution to the differential equation in (1), we assume the following.

*Assumption 1: The function  $f : X \rightarrow \mathbb{R}^n$  is locally Lipschitz in  $X$ , and the solution of the differential equation in (1) is well-defined  $\forall t \geq 0$ .*  $\square$

To prove Theorem 1 we also need the following assumption.

*Assumption 2: The pair  $(A, C)$  is observable.*  $\square$

### B. A parameterized Luenberger observer

We rely on a simple generalization of the well-known Luenberger observer, originally proposed for estimating the state variables of dynamic linear systems [14]. The proposed observer for system (1) has the following form:

$$\dot{\hat{x}} = A\hat{x} + f(\hat{x}) + L(y - C\hat{x}) + \gamma(y - C\hat{x}, w), \quad t \geq 0, \quad (2)$$

where  $\hat{x}(t) \in \mathbb{R}^n$  is the estimate of the vector  $x(t)$  at time  $t$ ,  $L \in \mathbb{R}^{n \times m}$  is a matrix, and  $\gamma : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}^n$  is a parameterized function that depends on the vector of parameters denoted by  $w \in \mathbb{R}^p$ . In Section V, we shall discuss the choice of a class of functions  $\gamma$  that exhibit properties to be exploited in the design of the estimator. Let  $Z \subset \mathbb{R}^m$  be the compact set to which  $z \triangleq y - C\hat{x}$  belongs for  $y = Cx$ ,  $x \in X$ .

The following assumption defines admissible functions  $\gamma$ .

*Assumption 3: For every  $w \in \mathbb{R}^p$ , the function  $\gamma(\cdot, w) : Z \rightarrow \mathbb{R}^n$  is locally Lipschitz in  $Z$  and such that  $\gamma(0, w) = 0$ . Moreover, the solution of the differential equation (2) is well-defined  $\forall t \geq 0$ .*  $\square$

The Lipschitz condition in Assumption 3 allows to ensure that there exists a unique local solution to the differential equation (2) describing the observer. The condition  $\gamma(0, w) = 0$ ,  $\forall w \in \mathbb{R}^p$ , guarantees that the origin is an equilibrium point for the dynamic equation of the estimation error, which is obtained from (1) and (2):

$$\dot{e} = (A - LC)e + f(x) - f(\hat{x}) - \gamma(Ce, w), \quad t \geq 0, \quad (3)$$

Moreover, under the condition  $\gamma(0, w) = 0$ , if there exists  $T \geq 0$  such that  $\hat{x}(T) = x(T)$ , then  $\hat{x}(t) = x(t)$  for  $t \geq T$ .

Finally, note that the last part of Assumption 3 enables to admit the existence of a regime for the dynamics of the estimation error, and thus it makes sense to study its asymptotic behavior, as pointed out in Assumption 1.

### C. Stability of the estimation error

The dynamics of the estimation vector  $\hat{x}$  is completely determined by equation (2) and by the initial conditions, but the choice of the matrix  $L$  and the vector  $w$  may be made in such a way to ensure the asymptotic stability of the estimation error. To this purpose, we shall look for a Lyapunov function whose time derivative is negative definite when computed on the trajectories of the estimation error. Thanks to the smoothness of  $f$  (see Assumption 1), the structure (2) chosen for the estimator, and the hypotheses on  $\gamma$  (see Assumption 3), we shall obtain the negative definiteness

of the derivative of the Lyapunov function on the whole set  $X \times \{E \setminus \{0\}\}$  by imposing it only on a finite grid of points, provided that such a grid is suitably chosen. A result presented in [9] enables to conclude about the stability of the estimation error as follows.

*Theorem 1: Consider the state observer (2) for the system (1) and suppose that Assumptions 1, 2, and 3 hold. Let  $E \subset \mathbb{R}^n$  be a connected, compact set to which the estimation error belongs,  $M$  a positive integer, and  $S_M \subset S \triangleq X \times \{E \setminus \{0\}\}$  a finite set of points  $s_i \triangleq \text{col}(x_i, e_i)$ ,  $i = 1, 2, \dots, M$ , corresponding to the first  $M$  points of a given sequence  $\{s_i\}$ . Let  $V(e) = e^T P e$  a Lyapunov function, where  $P \in \mathbb{R}^{n \times n}$  is a symmetric, positive definite matrix that satisfies the inequality*

$$(A - LC)^T P + P(A - LC) < 0, \quad (4)$$

where  $L$  is the gain matrix. Moreover, let  $\dot{V} \triangleq \frac{dV}{dt}$  be the derivative of  $V$  and suppose there exists a vector  $w^* \in \mathbb{R}^p$  such that

$$\epsilon_{\dot{V}} \triangleq - \max_{\bar{s} \in S_M} \dot{V}(\bar{s}, w^*) > 0. \quad (5)$$

Finally, suppose the set  $S_M$  is such that

$$\theta(S_M) < \frac{\epsilon_{\dot{V}}}{\lambda_{\dot{V}}}, \quad (6)$$

where  $\lambda_{\dot{V}}$  is the Lipschitz constant of the function  $\dot{V}(s, w)$  for  $w = w^*$ , i.e.,  $|\dot{V}(s, w^*) - \dot{V}(s', w^*)| \leq \lambda_{\dot{V}} \|s - s'\|$ ,  $\forall s, s' \in S$ , and

$$\theta(S_M) \triangleq \sup_{s \in S} \min_{\bar{s} \in S_M} \|s - \bar{s}\|. \quad (7)$$

Then the estimation error  $e(t) = x(t) - \hat{x}(t)$  of the estimator (2) converges asymptotically to zero for any  $e(0) \in E$ .  $\square$

To simplify the notation, we do not consider some dependences, as  $\epsilon_{\dot{V}}$  depends on the number  $M$  of sample points and on the number  $\nu$  of basis functions in the parameterized innovation function, as well as  $\lambda_{\dot{V}}$  depends on  $\nu$ .

The solution of the matrix inequality (4) can be obtained, after a change of variables, by a standard LMI routine [15].

### D. The design of the observer

Following Theorem 1, an approach to the selection of the parameter vector  $w^* \in \mathbb{R}^p$  was proposed in [9], based on a penalty-function method [16, pp. 390-398]. Specifically, if a sufficiently large number of sampling points is chosen, one can try to satisfy condition (15) together with the determination of  $w^*$  such that (5) is fulfilled by solving the following nonlinear programming problem:

$$w^* \in \underset{w \in \mathbb{R}^p}{\text{argmin}} J_M(w), \quad (8)$$

where, for a fixed  $\Delta > 0$ , the cost function is given by

$$J_M(w) = \sum_{s_i \in S_M} \left\{ \max \left[ 0, \dot{V}(s_i, w) + \Delta \right] \right\}^2. \quad (9)$$

By its definition,  $J_M(w) \geq 0$  for every  $w \in \mathbb{R}^p$  and, by inspection of the proof of Theorem 1, the condition (5) implies that there exists  $w^* \in \mathbb{R}^p$  such that  $\dot{V}(s, w^*) < 0, \forall s \in S$ . Hence, by taking  $\Delta = \min_{s_i \in S_M} |\dot{V}(s_i, w^*)|$ , one has  $J_M(w^*) = 0$ , and so  $w^* \in \underset{w \in \mathbb{R}^p}{\operatorname{argmin}} J_M(w)$ . On the other hand, if  $\bar{w} \in \underset{w \in \mathbb{R}^p}{\operatorname{argmin}} J_M(w)$ , then  $\dot{V}(s_i, \bar{w}) < 0, \forall s_i \in S_M$ , hence in Theorem 1 one can take  $w^* = \bar{w}$ . In Section V, we shall consider certain parameterized innovation functions, including some commonly used neural networks, whose choice allows one to satisfy condition (5), hence (on the basis of the discussion reported above) to guarantee that  $\underset{w \in \mathbb{R}^p}{\operatorname{argmin}} J_M(w)$  is nonempty.

Summing up, the design of the observer is reduced to the solution of the optimization problem (8). The solution can be obtained via nonlinear programming techniques, which allow one to find *off line* the values of the parameter vector  $w^*$ . Clearly, the use of efficient sampling techniques allows to reduce the computational burden involved in (8). To this end, a discussion on such issues is given in Section IV.

### III. DESIGN OF CONTROLLERS WITH GUARANTEED STABILITY

#### A. System description

In this section, we address the problem of designing a controller to stabilize a system of the form

$$\dot{x} = Ax + Bu + f(x) \quad (10)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector (supposed measurable) and  $u(t) \in \mathbb{R}^m$  is the control vector.  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ , and  $f : X \rightarrow \mathbb{R}^n$  are known.

*Assumption 4:* The pair  $(A, B)$  is controllable.  $\square$

*Assumption 5:* The function  $f : X \rightarrow \mathbb{R}^n$  is locally Lipschitz in  $X$  and such that  $f(0) = 0$ .  $\square$

For the purpose of stabilizing the system (10), we define the following control function:

$$u = -Kx + \gamma(x, w) \quad (11)$$

where  $K \in \mathbb{R}^{m \times n}$  is a gain matrix and  $\gamma : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^m$  is a parameterized function depending on a vector of parameters  $w \in \mathbb{R}^p$ . After replacing equation (11) in (10), we rewrite the system in the form

$$\dot{x} = (A - BK)x + f(x) + \gamma(x, w) \quad (12)$$

We introduce the following

*Assumption 6:* For every  $w \in \mathbb{R}^p$ , the function  $\gamma(\cdot, w) : Z \rightarrow \mathbb{R}^m$  is locally Lipschitz in  $Z$  and such that  $\gamma(0, w) = 0$ . Moreover, the solution of the differential equation (12) is well-defined  $\forall t \geq 0$ .  $\square$

It can be easily verified that under these conditions there exists a unique local solution to the differential equation (12) and that  $\bar{x} = 0$  is an equilibrium point (see, e.g., [17]). Therefore, the problem consists in finding the matrix  $K$  and the vector  $w$  so that the system (12) is asymptotically stable.

#### B. The design of the controller

Likewise in the design of the observer, the proposed approach for the stabilization of the system (12) is based on determining a matrix  $K$  and a vector  $w$  that lead to a negative time derivative of a quadratic Lyapunov function on  $X$ .

Actually, it can be shown that, on the basis of the regularity assumptions concerning the functions  $f$  and  $\gamma$ , it is sufficient that  $\dot{V}$  is negative only on a finite sample of points suitably chosen in the state space.

Now, we can state the following.

*Theorem 2:* Consider the controller (11) for the system (10) and suppose that Assumptions 4, 5, and 6 hold. Let  $X \subset \mathbb{R}^n$  be a connected, compact set to which the state belongs,  $M$  a positive integer, and  $X_M \subset X \setminus \{0\}$  a finite set of points  $x_i, i = 1, 2, \dots, M$ , corresponding to the first  $M$  points of a given sequence  $\{x_i\}$ . Let  $V(x) = x^T P x$  a Lyapunov function, where  $P \in \mathbb{R}^{n \times n}$  is a symmetric, positive definite matrix that satisfies the inequality

$$(A - BK)^T P + P(A - BK) < 0, \quad (13)$$

where  $K$  is the gain matrix. Moreover, let  $\dot{V} \triangleq \frac{dV}{dt}$  be the derivative of  $V$  and suppose there exists a vector  $w^* \in \mathbb{R}^p$  such that

$$\epsilon_{\dot{V}} \triangleq - \max_{\bar{x} \in X_M} \dot{V}(\bar{x}, w^*) > 0. \quad (14)$$

Finally, suppose the set  $X_M$  is such that

$$\theta(X_M) < \frac{\epsilon_{\dot{V}}}{\lambda_{\dot{V}}}, \quad (15)$$

where  $\lambda_{\dot{V}}$  is the Lipschitz constant of the function  $\dot{V}(x, w)$  for  $w = w^*$ , i.e.,  $|\dot{V}(x, w^*) - \dot{V}(x', w^*)| \leq \lambda_{\dot{V}} \|x - x'\|, \forall x, x' \in X$ , and

$$\theta(X_M) \triangleq \sup_{x \in X} \min_{\bar{x} \in X_M} \|x - \bar{x}\|. \quad (16)$$

Then the state vector  $x(t)$  converges asymptotically to zero for any  $x(0) \in X$ .  $\square$

The proof of Theorem 2 follows similar steps as the one of Theorem 1, which can be found in [9].

The design of the controller is reduced to finding a vector  $w^*$  such that it satisfies (14). Also in this case, the solution of (13) can be obtained via LMIs.

### IV. QUASI-MONTE CARLO SAMPLING

The quantity  $\theta(S_M)$  and  $\theta(X_M)$  in Theorems 1 and 2, respectively, is usually called the *dispersion* of the sequence of  $M$  points of the set  $S_M$  or  $X_M$  [18], and it measures the uniformity of the distribution of the points in  $S$  or  $X$ , respectively.

In practice, points that are spread “in a uniform way,” i.e., without leaving regions of the space “undersampled,” and in such a way that the points are “close enough” to one another, correspond to a small value of  $\theta$ .

Various methods have been proposed in the literature to generate well-uniformly scattered deterministic sets as finite

portions of suitably generated infinite sequences, such as the *Niederreiter sequence*, the *Halton sequence*, the *Hammersley sequence*, the *Sobol' sequence*, the *Faure sequence* [18], [19]. In [18], a common general framework is presented for the construction of sequences of this kind, sometimes informally referred to as *low-discrepancy sequences*. In particular  $(t, n)$ -sequences, that generalize many of the aforementioned techniques, are discussed, together with the most relevant theoretical properties.

For the purposes of the method described in this work, it can be shown that it is possible to construct  $(t, n)$ -sequences  $\{\sigma_M\}$  that satisfy deterministically

$$\theta(\{\sigma_M\}) \leq \sqrt{n}O(M^{-1/n}(2 \log M)).$$

(with a little notational abuse, we have considered the sequence  $\{\sigma_M\}$  as a set). Thus it is guaranteed that, for every  $\epsilon_{\tilde{V}}$  and  $\lambda_{\tilde{V}}$ , (6) and (15) can be satisfied, i.e., there exists a number  $\tilde{M}$  of points such that  $\theta < \frac{\epsilon_{\tilde{V}}}{\lambda_{\tilde{V}}}$  for  $M > \tilde{M}$ .

When compared to random sampling, discretization based on low-discrepancy sequences generally suffers less from the formation of clusters of points in particular regions of the space, which undermines the sampling uniformity. Furthermore the discrepancy of i.i.d. uniform sampling has a *quadratic* rate of convergence, typical of Monte Carlo methods, in terms of number of points  $M$  [19].

For further comparisons between random and low-discrepancy sequences and a discussion on the use of quasi-Monte Carlo sampling with neural network architectures to solve problems of function approximation and approximate Dynamic Programming, see [11], [12], [13].

## V. USE OF PARAMETERIZED MAPPINGS

We further restrict the classes (2) of observers and (11) of controllers.

*Definition 1:* Let  $\Lambda \subset \mathbb{R}^m$  be compact,  $\nu, l_i, i = 1, \dots, \nu$  be positive integers,  $\varphi_i : \Lambda \times \mathbb{R}^{l_i} \rightarrow \mathbb{R}$ , and  $\mathcal{N}(\nu) = \sum_{i=1}^{\nu} l_i + n\nu$ . We call one-hidden-layer networks of order  $\nu$  the functions belonging to the set

$$A_\nu \triangleq \{ \gamma_\nu : \Lambda \times \mathbb{R}^{\mathcal{N}(\nu)} \rightarrow \mathbb{R}^n \text{ such that}$$

$$(i) \gamma_{\nu j}(\xi, \omega_{\nu j}) = \sum_{i=1}^{\nu} c_{ij} \varphi_i(\xi, \kappa_i), \exists \bar{c} \in \mathbb{R}^+ \text{ such that}$$

$$|c_{ij}| \leq \bar{c}, \kappa_i \in \mathbb{R}^{l_i}, i = 1, \dots, \nu, j = 1, \dots, n, \omega_{\nu j} \triangleq \text{col}(c_{ij}, \kappa_i : i = 1, \dots, \nu);$$

(ii) the functions  $\varphi_i(\cdot, \kappa_i)$  are Lipschitz, i.e.,  $\forall i = 1, \dots, \nu \exists \lambda_i \in \mathbb{R}^+$  such that  $\forall \kappa_i \in \mathbb{R}^{l_i}, |\varphi_i(\xi, \kappa_i) - \varphi_i(\xi', \kappa_i)| \leq \lambda_i |\xi - \xi'|$ ;

(iii) the functions  $\varphi_i(\cdot, \kappa_i)$  are bounded in aggregate, i.e.,  $\exists \tilde{c} \in \mathbb{R}^+$  such that  $\forall i = 1, \dots, \nu, \forall \kappa_i \in \mathbb{R}^{l_i}, \max_{\xi \in K} |\varphi_i(\xi, \kappa_i)| \leq \tilde{c}$ ;

(iv)  $\bigcup_{\nu=1}^{\infty} A_\nu$  is dense in  $C(\Lambda, \mathbb{R}^m)$  wrt the sup norm. } where  $C(\Lambda, \mathbb{R}^n)$  is the space of continuous  $n$ -valued functions on  $\Lambda$  with the sup norm.

Using a one-hidden-layer network  $\gamma_\nu \in A_\nu$  as a parameterized innovation function in (2) and (11), we get

$$\dot{\hat{x}} = A\hat{x} + f(\hat{x}) + L(y - C\hat{x}) + \gamma_\nu(y - C\hat{x}, w_\nu), \quad (17)$$

$$u = -Kx + \gamma_\nu(x, w_\nu) t \geq 0. \quad (18)$$

Item (ii) in Definition 1 takes into account Assumption 6. To satisfy Assumption 3, we also impose

$$\varphi_i(\cdot, \kappa_i) = 0 \quad i = 1, \dots, \nu \quad \forall \kappa_i \in \mathbb{R}^{l_i}. \quad (19)$$

The functions  $\gamma_\nu$  are Lipschitz, being the sums of  $\nu$  Lipschitz functions bounded by  $\bar{c}\tilde{c}$ . Then, we have the following proposition.

*Proposition 1:* For any positive integer  $\nu$ , one-hidden-layer networks of order  $\nu$  are admissible functions  $\gamma$  in (11) and, if also (19) holds, in (2).  $\square$

Feedforward neural networks with suitable choices of the basis functions satisfy Definition 1, and will be used in the numerical simulations of Section VI. For the properties of approximators with such a structure and their application to optimization problems, see [20], [21], [22], [23], [24].

## VI. NUMERICAL RESULTS

### A. Neural observer

Let us consider a double pendulum equation with the state vector  $x = (x_1, x_2, x_3, x_4)^T$  to be estimated by using only the measurements of  $x_1$  and  $x_2$

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= -\frac{m_2^2 l_2 \sin(x_1 - x_2) \cos^2(x_1 - x_2) x_4^2}{M^2 l_1 - M l_1 m_2 \cos^2(x_1 - x_2)} \\ &\quad - \frac{m_2 g \sin(x_1) \cos^2(x_1 - x_2)}{M l_1 - m_2 l_1 \cos^2(x_1 - x_2)} \\ &\quad - \frac{m_2 x_3^2 \sin(x_1 - x_2) \cos(x_1 - x_2)}{M - m_2 \cos^2(x_1 - x_2)} \\ &\quad - \frac{m_2 g \sin(x_2) \cos(x_1 - x_2)}{M l_1 - m_2 l_1 \cos^2(x_1 - x_2)} \\ &\quad - \frac{m_2 l_2 x_4^2 \sin(x_1 - x_2)}{M l_1} - \frac{g \sin(x_1)}{l_1} \\ \dot{x}_4 &= \frac{m_2 \sin(x_1 - x_2) \cos(x_1 - x_2) x_4^2}{M - m_2 \cos^2(x_1 - x_2)} \\ &\quad + \frac{M g \sin(x_1) \cos(x_1 - x_2)}{M l_2 - m_2 l_2 \cos^2(x_1 - x_2)} \\ &\quad + \frac{M l_1 x_3^2 \sin(x_1 - x_2)}{M l_2 - m_2 l_2 \cos^2(x_1 - x_2)} \\ &\quad - \frac{M g \sin(x_2)}{M l_2 - m_2 l_2 \cos^2(x_1 - x_2)} \\ y_1 &= x_1 \\ y_2 &= x_2 \end{aligned} \quad (20)$$

where  $M = 2 \text{ Kg}$ ,  $m_2 = 1 \text{ Kg}$ ,  $l_1 = l_2 = 1 \text{ m}$ , and  $g = 9.8 \text{ m/s}^2$ .

The dynamics of (20) is not asymptotically stable since no dissipative forces were considered. Referring to (1), from (20) we have

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Note that the pair  $(A, C)$  is observable.

The design of observer (17) for (20) requires first the selection of a suitable class of one-hidden-layer networks, according to the requisites of Section V. To this purpose, we chose a one hidden-layer feedforward neural network with  $\nu = 10$  sigmoidal activation functions, belonging to the set  $A_\nu$  defined in Section V. We found

$$L = \begin{pmatrix} 4.5017 & 1.269 \\ -2.2645 & 0.6149 \\ 4.9862 & 3.3717 \\ -3.1185 & 1.2539 \end{pmatrix}$$

$$P = \begin{pmatrix} 1.814 & 1.3368 & -0.5 & 0.6881 \\ 1.3368 & 2.8471 & -0.6881 & -0.5 \\ -0.5 & -0.6881 & 0.6328 & 0.2080 \\ 0.6881 & -0.5 & 0.208 & 1.2602 \end{pmatrix}$$

which satisfy (4). The optimal parameters of the neural network were determined by minimizing the cost (9) on a discretization of the set  $X \times \{E \setminus \{0\}\}$ , where  $X = [-\pi, \pi] \times [-4\pi, 4\pi] \times [-\pi/2, \pi/2] \times [-2\pi, 2\pi]$  and  $E = [-\pi, \pi] \times [-\pi, \pi] \times [-2\pi, 2\pi] \times [-2\pi, 2\pi]$ , that is made of 2000 points resulting from a low-discrepancy sequence (specifically, a Sobol' sequence [25]). The minimization was performed using a standard Matlab routine of the Optimization Toolbox [26].

A comparison with the EKF is provided. We have considered uncertainties in the initial state and disturbances acting on the dynamics and measurement equations modelled as additive, truncated, Gaussian, zero-mean random noises with covariance matrices equal to  $0.1I$ ,  $0.001I$ , and  $0.01I$ .

The results obtained for different initial conditions are shown in Fig. 1. Note that the EKF is affected by a large overshooting in the transient, while this does not occur for the neural estimator.

### B. Neural controller

This example concerns a Lorenz attractor, which is characterized by chaotic behavior and dynamics represented by

$$\begin{aligned} \dot{x}_1 &= 10(x_2 - x_1) + u \\ \dot{x}_2 &= 28x_1 - x_2 - x_1x_3 \\ \dot{x}_3 &= -\frac{8}{3}x_3 + x_1x_2. \end{aligned} \quad (21)$$

The above system shows a strange attractor if  $u = 0$ . We reformulate equation (21) as in (10) by defining

$$A = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The pair  $(A, B)$  is controllable. We found

$$K = (-8 \quad 12 \quad 1)$$

$$P = \begin{pmatrix} 2.4224 & 0.1552 & -0.1809 \\ 0.1552 & 0.1897 & 0.0564 \\ -0.1809 & 0.0564 & 0.2553 \end{pmatrix}$$

which satisfy (13). We have again chosen a one-hidden layer feedforward neural network with  $\nu = 10$  sigmoidal activation functions, to be employed as the function  $\gamma \in A_\nu$ . In order to minimize the cost (9), we have employed a discretization of  $X = [-13, 13] \times [-13, 13] \times [-13, 13]$  made of 4000 points from a Sobol' sequence.

In the simulations shown in Fig. 2, truncated Gaussian, zero-mean noises were added to the dynamics of the system with covariance  $Q$ . To evaluate the effectiveness of the approach, the neural controller is compared with a standard linear controller applied to the linearized system. Note that the neural controller exhibits a faster speed of convergence than that of the linear one.

### REFERENCES

- [1] F. E. Thau, "Observing the state of non-linear dynamic systems", *Int. J. of Control*, vol. 17, no. 3, pp. 471–479, 1973.
- [2] S. R. Kou, D. L. Elliott, and T. J. Tarn, "Exponential observers for nonlinear dynamic systems", *Information and Control*, vol. 29, pp. 204–216, 1975.
- [3] A. J. Krener and A. Isidori, "Linearization by output injection and nonlinear observers", *Systems & Control Letters*, vol. 3, pp. 47–52, 1983.
- [4] A. J. Krener and W. Respondek, "Nonlinear observer with linearizable error dynamics", *SIAM J. on Control and Optimization*, vol. 23, no. 2, pp. 197–216, 1985.
- [5] J. P. Gauthier, H. Hammouri, and S. Othman, "A simple observer for nonlinear systems applications to bioreactors", *IEEE Trans. on Automatic Control*, vol. 37, no. 6, pp. 875–880, 1992.
- [6] B. L. Walcott and S. H. Zak, "State observation of nonlinear uncertain dynamical systems", *IEEE Trans. on Automatic Control*, vol. 32, no. 2, pp. 166–170, 1987.
- [7] S. Yu and A. M. Annaswamy, "Stable neural controllers for nonlinear dynamic systems", *Automatica*, vol. 34, no. 5, pp. 641–650, 1998.
- [8] T. A. Johansen, "Computation of Lyapunov functions for smooth nonlinear systems using convex optimization", *Automatica*, vol. 36, no. 11, pp. 1617–1626, 2000.
- [9] A. Alessandri, C. Cervellera, and M. Sanguineti, "Design of asymptotic state observers for nonlinear systems: an approach based on neural networks and nonlinear programming", *IEEE Transactions on Neural Networks*, to appear.
- [10] A. Alessandri, C. Cervellera, and M. Sanguineti, "Functional optimal estimation problems and their solution by nonlinear approximation schemes", *Journal of Optimization Theory and Applications*, to appear.
- [11] C. Cervellera and M. Muselli, "Deterministic design for neural network learning: An approach based on discrepancy", *IEEE Trans. on Neural Networks*, vol. 15, no. 3, pp. 533–544, 2004.
- [12] C. Cervellera, V.C. Chen, and A. Wen, "Optimization of a large-scale water reservoir network by stochastic dynamic programming with efficient state space discretization", *European Journal of Operational Research*, vol. 171, no. 3, pp. 1139–1151, 2006.
- [13] C. Cervellera, V.C. Chen, and A. Wen, "Neural network and regression spline value function approximations for stochastic dynamic programming", *Computers and Operations Research*, to appear.
- [14] D. G. Luenberger, "Observing the state of a linear system", *IEEE Trans. Mil. Electron.*, vol. 8, pp. 74–80, 1964.
- [15] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, vol. 15 of *Studies in Applied Mathematics*, SIAM, Philadelphia, PA, 1994.

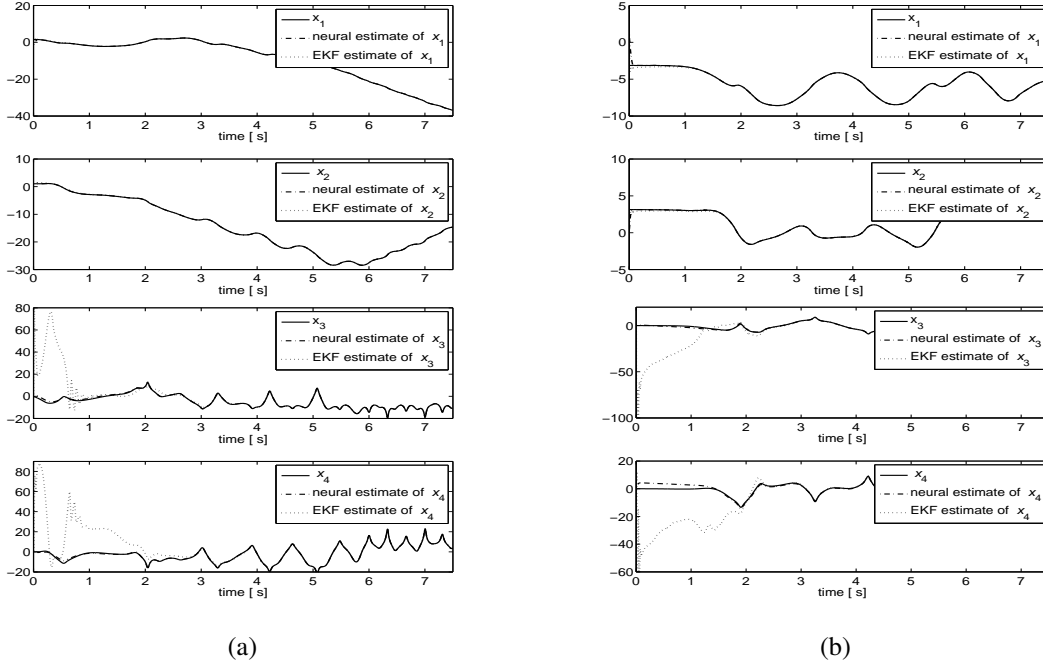


Fig. 1. Observer: numerical results for (a)  $x(0) = [-\pi/2, -\pi/3, 0, 0]^T$ ,  $\hat{x}(0) = [-1, 1, -1, 1]^T$  and (b)  $x(0) = [-\pi/2, -\pi/3, 0, 0]^T$ ,  $\hat{x}(0) = [-1, 1, -1, 1]^T$ .

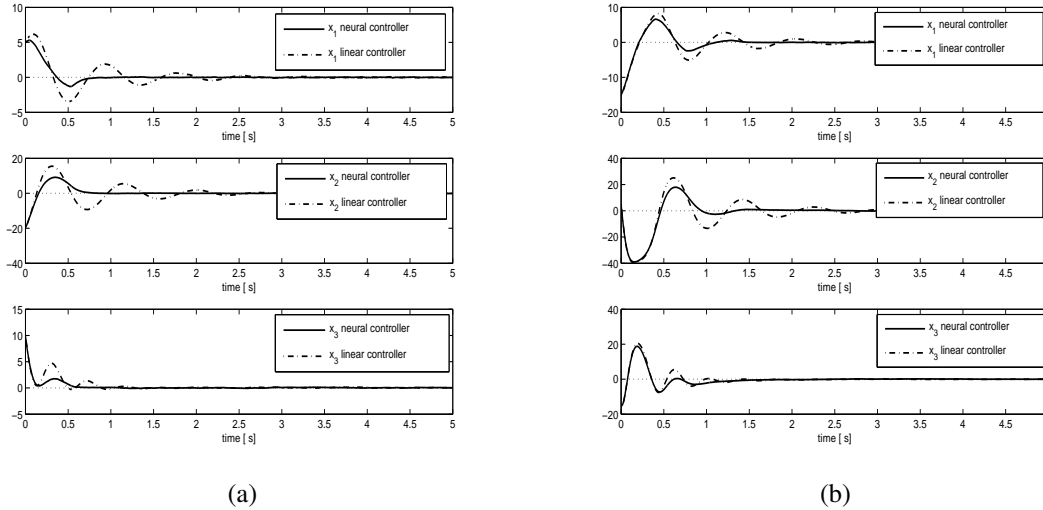


Fig. 2. Controller: numerical results for (a)  $x(0) = [5, -20, 10]^T$ ,  $Q = 0.01I$  and (b)  $x(0) = [-15, 5, -15]^T$ ,  $Q = 0.05I$ .

[16] D. P. Bertsekas, *Nonlinear Programming*, Athena Scientific, Belmont, MA, 1999.

[17] H. K. Khalil, *Nonlinear Systems*, Prentice Hall, Upper Saddle River, NJ, 1996.

[18] H. Niederreiter, *Random Number Generation and Quasi-Monte Carlo Methods*, SIAM, Philadelphia, 1992.

[19] K.-T. Fang and Y. Wang, *Number-theoretic Methods in Statistics*, Chapman & Hall, London, 1994.

[20] V. Kurková and M. Sanguineti, “Bounds on rates of variable-basis and neural-network approximation”, *IEEE Trans. on Information Theory*, vol. 47, no. 6, pp. 2659–2665, 2001.

[21] V. Kurková and M. Sanguineti, “Comparison of worst case errors in linear and neural network approximation”, *IEEE Trans. on Information Theory*, vol. 48, no. 1, pp. 264–275, 2002.

[22] P. Kainen, V. Kurková, and M. Sanguineti, “Minimization of error functionals over variable-basis functions”, *SIAM J. on Optimization*, vol. 14, pp. 732–742, 2003.

[23] V. Kurková and M. Sanguineti, “Error estimates for approximate optimization by the extended Ritz method”, *SIAM J. on Optimization*, vol. 15, pp. 461–487, 2005.

[24] V. Kurková and M. Sanguineti, “Learning with generalization capability by kernel methods of bounded complexity”, *J. of Complexity*, vol. 21, no. 3, pp. 350–367, 2005.

[25] I. M. Sobol’, “The distribution of points in a cube and the approximate evaluation of integrals”, *Zh. Vychisl. Mat. i Mat. Fiz.*, vol. 7, pp. 784–802, 1967.

[26] T. Coleman, M. A. Branch, and A. Grace, *Optimization Toolbox User’s Guide*, The Math Works Inc., Natick, MA, 1999.