Connections between $L_p$ stability and asymptotic stability of nonlinear switched systems

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Abstract

Here are considered nonlinear switched systems in which the switching occurs among a class of subsystems that are characterized by input–output properties stated in terms of $L_p$ spaces of signals. The relationships between the $L_p$ stability of each subsystem and the internal stability of the switched system are studied. In particular, conditions on the dwell time of the switching signals that guarantee the asymptotic stability of the overall system are provided. The connections among these conditions and the $L_p$ input–output properties of the subsystems are investigated.

Keywords: Switched systems; Asymptotic stability; $L_p$ stability; $L_p$ optimal estimation problems; $L_p$ reachability and observability; Dwell time

1. Introduction

Switched systems have gained a lot of attention, as they are well-suited to dealing with problems where different dynamic behaviours are combined together (e.g., continuous-time dynamics, discrete-time dynamics, logic decisions). The interaction among such different abstract modelling levels often makes difficult the stability analysis of these systems, and motivates the intensive investigations in this area.

The interest in the stability properties of switched systems has been particularly focused on finding conditions on the switching signals and on the subsystems, in order to guarantee the stability of the overall system. A general introduction to the issues related to the stability of switched systems is given in [1]. Most works on this subject are based on the Lyapunov stability formulation (see, for example, [2]), which, in the case of linear switched systems, provides a systematic solution framework exploiting well-developed design tools such as the Linear Matrix Inequality (LMI) approach [3]. However, other paradigms have been proposed in alternative or support to Lyapunov stability. They are basically classified into two families, usually referred to as “internal stability” and “external stability” or “input–output stability” [4].

Internal stability requires the system state to tend to zero from a nonzero initial value when no input is present, and to remain bounded when a bounded input acts on the system. Such a type of stability was studied via the so-called ISS

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The concept of input–output stability naturally arises if one associates with the dynamic system a mapping that assigns to each input a corresponding output. The stability of this mapping can be investigated in function spaces of signals [4, Chapter 6]. The connections between these concepts of stability rely on the detectability properties of the system [7,8].

An important family of function spaces in input/output stability are \( L_p \) spaces, \( 1 \leq p \leq \infty \). The role played by \( L_p \) stability goes beyond its relationship with Lyapunov’s theory (e.g., the notion of \( L_2 \)-gain in defining the passivity to deal with nonlinear control problems [9]). In statistics, the interest in \( L_p \) spaces with \( p = 1 \) or slightly greater than 1 is motivated by the fact, that such \( L_p \)-norms are more robust than the \( L_2 \)-norm [10]. Moreover, input–output stability results in function spaces provide a solid mathematical basis to address the design of optimal state estimators [11,12].

For example, in [11] we investigated \( L_p \) stability for the dynamics of the estimation error for a class of continuous-time, nonlinear dynamic systems and a corresponding class of estimators designed according to an \( L_p \) optimality criterion.

In the case of linear dynamic systems without switching, well-established relationships between internal and \( L_p \) stability are available since the late eighties [13]. Much less is known about the connections between these two concepts of stability for nonlinear systems, even in the non-switched case. The first results reported in [14] about the possibility of finding Lyapunov functions for input–output stable systems were extended in [15], where conditions under which finite-gain \( L_p \) stability implies local asymptotic stability, and global exponential stability implies finite-gain stability were given. In [16] a local notion of input–output stability, called “small-signal \( L_p \) stability”, was defined, and its relationships with asymptotic stability were investigated. Later, the concept of small-signal \( L_p \) stability was extended by the notion of “gain over a set”, whose connections with local Lyapunov stability were studied in [7].

A concept of input–output stability, called “\( W \) stability” and based on the use of Sobolev spaces of input and output signals, was proposed in [17]. Its relationships with asymptotic stability and \( L_p \) stability were addressed in [17,18]. “Local \( W \) stability” is close to, but not coincident with, the small-signal \( L_p \) stability introduced in [16]. In [19], the so-called “\( L_p \)-input converging state property” was investigated for nonlinear systems and conditions were given under which an input of bounded energy originates a converging state.

As to internal stability for switched systems, the existence of a Lyapunov function common to all subsystems is necessary and sufficient for asymptotic stability under an arbitrary switching law [20]. Switched systems arising in applications often do not enjoy a common Lyapunov function; nevertheless, they may be asymptotically stable under conditions on the switching [21]. The quadratic stability of nonlinear switched systems (which, unlike for linear systems, is not equivalent to asymptotic stability), was investigated for piecewise linear systems (see [22,23] and the references therein) and for some classes of nonlinear systems (see, e.g., [24,25]).

The focus of this work is on the connections between internal and \( L_p \) stability, for a class of nonlinear continuous-time switched systems. First, by exploiting the finite-gain \( L_p \) stability over a set, the \( L_p \) reachability and the \( L_p \) observability [7] of the mapping associated with the subsystems among which the switching occurs, we guarantee the existence, for each subsystem, of two Lyapunov functions with certain properties. Then, by combining these properties with a result from [26], we prove the asymptotic stability of the overall unforced switched system, provided that the average dwell time of the switching signals is suitably bounded from below. We show that such a lower bound depends on the properties of the above-mentioned Lyapunov functions related to the \( L_p \) stability, reachability and observability of the subsystems.

The outline of the paper is the following. Section 2 introduces the main concepts, gives the definitions that will be used throughout the paper, and contains some mathematical preliminaries. The main results are stated and proved in Section 3. Section 4 contains some conclusive remarks.

2. Mathematical framework and definitions

We consider the following representation of a switched system:

\[
\Sigma. \begin{cases}
\dot{x}(t) = f_s(x(t), u(t)) \\
y(t) = g_s(x(t), u(t)),
\end{cases}
\]

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( u(t) \in \mathbb{R}^m \) is the control (input) vector, \( y(t) \in \mathbb{R}^p \) is the output vector, \( s \) is a switching signal, i.e., a piecewise constant function \( s: [0, +\infty) \rightarrow \Gamma \), where \( \Gamma \triangleq \{\sigma_i, i = 0, \ldots, \} \) is an index set, and \( f_s, g_s : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R} \).
Each \( \sigma \in \Gamma \) corresponds to a subsystem, which becomes “active” during one or more intervals of time, according to the switching signal \( s \). More specifically, associated with each signal \( s \), there exist a sequence of real numbers \( 0 = t_0 < t_1 < \cdots < t_k < \cdots \) and a sequence of indexes \( \sigma_0, \sigma_1, \ldots, \sigma_k, \ldots \), such that \( s(t) = \sigma_k \) for \( t_k \leq t < t_{k+1} \). Correspondingly, for \( t_k \leq t < t_{k+1} \) one has \( f_s = f_{\sigma_k} \) and \( g_s = g_{\sigma_k} \). The generic subsystem \( \sigma \) is described by:

\[
\begin{align*}
\sigma: \{ & \dot{x}(t) = f_\sigma(x(t), u(t)) \\
& y(t) = g_\sigma(x(t), u(t)) \}
\end{align*}
\]

(2)

In the following, the subsystem associated with the index \( \sigma \) will be called “subsystem \( \sigma \)”.

The admissible control signals \( u \) belong to the space \( L^m_{\infty} \), which is the set of all Lebesgue-measurable functions \( u : [0, \infty) \to \mathbb{R}^m \) that are essentially bounded, i.e., such that \( \text{ess sup}_{t \geq 0} |u(t)| < \infty \), where “ess, sup” denotes the essential supremum (i.e., supremum except on sets of measure zero), and for every \( m \geq 1 \), \( \| \cdot \| \) denotes the Euclidean norm in \( \mathbb{R}^m \). The space \( L^m_{\infty} \) is a Banach space with the norm

\[
\|u\|_{\infty} \triangleq \text{ess sup}_{t \geq 0} |u(t)|.
\]

The space \( L^p_m \), for \( p \in [1, \infty) \) and \( m \in \mathbb{N}^+ \), is the Banach space of all Lebesgue measurable functions \( u : [0, \infty) \to \mathbb{R}^m \) with the \( L^p \)-norm, defined as:

\[
\|u\|_p \triangleq \left( \int_0^{\infty} \|u(t)\|^p \, dt \right)^{1/p}, \quad p \in [1, \infty).
\]

The space \( L^2_m \) is a Hilbert space with the inner product defined for every \( w, z \in L^2_m \) as \( \langle w, z \rangle = \int_0^\infty w(x)' z(x) \, dx \) (the symbol ‘’ denotes transposition), which induces the \( L^2 \)-norm.

We consider the following family of admissible right-hand sides for the switched system \( \Sigma \) (see (1)):

\[
\mathcal{P} \triangleq \{ f_\sigma : \sigma \in \Gamma \text{ such that } \forall \sigma \in \Gamma, f_\sigma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \text{ is locally Lipschitz} \}.
\]

Without any loss of generality, we suppose that there exists a one-to-one correspondence between the elements of \( \Gamma \) and those of \( \mathcal{P} \), i.e., for all \( \sigma, \sigma' \in \Gamma \) one has

\[
\sigma \neq \sigma' \Rightarrow f_\sigma \neq f_{\sigma'}.
\]

We also assume that \( g_s(0, 0) = f_s(0, 0) = 0 \) for every \( s \in S \); hence, \( x = 0 \) is an equilibrium point of the unforced system \( \dot{x}(t) = f_s(x(t), 0) \), for every \( s \in S \).

For every \( k = 0, 1, \ldots \), every \( f_{\sigma_k} \in \mathcal{P} \), every \( x(t_k) = x_k \in \mathbb{R}^n \), and every \( u \in L^m_{\infty} \), there exists a unique absolutely continuous function:

\[
\Theta_k : [t_k, t_{k+1}) \cap [t_k, T_{\sigma_k-x_k}) \to \mathbb{R}^n
\]

such that \( \Theta_k(t_k) = x_k \) and \( \dot{\Theta}_k(t) = f_{\sigma_k}(\Theta_k(t), u(t)) \) almost everywhere in \( [t_k, t_{k+1}) \cap [t_k, T_{\sigma_k-x_k}) \), where \( T_{\sigma_k-x_k} \) denotes its maximum interval of definition (see, e.g., [28]). Such a solution is called a trajectory of the subsystem associated with the index \( \sigma_k \in \Gamma \) and with the control \( u \in L^m_{\infty} \), and originating from \( x_k \in \mathbb{R}^n \); we denote it by \( x_{\sigma_k} (t, t_k, x_k, u) \). For every \( s \in S \), every \( x_0 \in \mathbb{R}^n \), and every \( u \in L^m_{\infty} \), a trajectory of the switched system is a locally absolutely continuous function:

\[
\Theta : [0, T_{s-x_0}) \to \mathbb{R}^n
\]

such that \( \Theta(t) = \Theta_k(t) \) for \( t \in [t_k, t_{k+1}) \cap [0, T_{s-x_0}) \), \( k = 0, 1, \ldots \), where \( T_{s-x_0} \) denotes the maximum interval of definition.

Finally, on the basis of the discussion reported above, we assume that the switched system \( \Sigma \) (see (1)) is forward complete.

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1 To avoid confusion, by \( | \cdot | \) we denote the Euclidean norm in \( \mathbb{R}^n \), and by \( \| \cdot \|_p \) we denote the \( L^p \)-norm of \( \mathbb{R}^m \)-valued functions, for any \( p \in [1, \infty) \) and \( m \geq 1 \).
As we are going to consider \( L_p \)-stable subsystems, we restrict our attention to input signals with finite \( L_p \)-norm; thus, \( u \in L_p^\infty \cap L_p^m \) for some \( 1 \leq p < \infty \). Recall that a function \( \varphi : [0, a) \to [0, +\infty) \) is a \( \mathcal{K} \)-class function if it is continuous, strictly monotone increasing, and such that \( \varphi(0) = 0 \). It is a \( \mathcal{K}_\infty \)-class function if \( a = +\infty \) and \( \lim_{r \to +\infty} \varphi(r) = +\infty \). Finally, \( \phi : [0, a) \times [0, +\infty) \to [0, +\infty) \) is a \( \mathcal{KL} \)-class function if, for a fixed \( s \), the mapping \( \phi(r, s) \) belongs to class \( \mathcal{K} \) with respect to \( r \) and, for a fixed \( r \), the mapping \( \phi(r, s) \) is decreasing with respect to \( s \) and \( \lim_{s \to +\infty} \phi(r, s) = 0 \).

**Definition 1 (\( L_p \) Uniform Reachability).** The subsystem corresponding to \( \sigma \in \Gamma \) is locally uniformly \( L_p \)-reachable if there exist a \( \mathcal{K} \)-class function \( \kappa_\sigma \) and a constant \( c_\sigma \in (0, +\infty) \) such that if \( |x| < c_\sigma \), then there exist \( u \in L_p^\infty \) and \( t_\sigma > 0 \) satisfying

(i) \( \|u\|_p \leq \kappa_\sigma (|x|) \),

(ii) \( \xi_\sigma (t_\sigma, 0, 0, u) = x \).

If this holds for all \( x \in \mathfrak{R}^n \) and \( \kappa_\sigma \) is a \( \mathcal{K}_\infty \) function, then the subsystem \( \sigma \) is uniformly \( L_p \)-reachable.

**Definition 2 (\( L_p \) Uniform Observability).** The subsystem corresponding to \( \sigma \in \Gamma \) is locally uniformly \( L_p \)-observable if there exist a \( \mathcal{K} \)-class function \( \zeta_\sigma \) and a constant \( c_\sigma \in (0, +\infty) \) such that:

\[
\text{if } |x| < c_\sigma, \quad \text{then } \|g_\sigma (\xi_\sigma (\cdot, 0, x, 0), 0)\|_p \geq \zeta_\sigma (|x|).
\]

If this holds for all \( x \in \mathfrak{R}^n \) and \( \zeta_\sigma \) is a \( \mathcal{K}_\infty \) function, then the subsystem \( \sigma \) is uniformly \( L_p \)-observable.

**Definition 3 (\( L_p \) Stability).** Let \( W \subseteq L_p^m \). The subsystem corresponding to \( \sigma \in \Gamma \) is uniformly finite gain \( L_p \)-stable over \( W \) if there exists a constant \( \gamma_{\sigma, W} \in (0, +\infty) \) such that

\[
\|y\|_p \leq \gamma_{\sigma, W} \|u\|_p \quad \forall u \in W,
\]

where \( \gamma(t) = g_\sigma (\xi_\sigma (t, 0, 0, u(t)), u(t)) \).

If \( W = L_p^m \), then the subsystem \( \sigma \) is finite-gain \( L_p \)-stable.

The quantity \( \gamma_{\sigma, W} \) is called \( L_p \)-gain over \( W \) of the subsystem corresponding to \( \sigma \in \Gamma \). If \( W = L_p^m \cap B_\infty (r) \), where \( B_\infty (r) \) denotes the ball of radius \( r \) in \( L_p^m \), then the Definition 3 coincides with the definition of small-signal \( L_p \) stability. In this paper we take \( W = B_p (r) \), where \( B_p (r) \) denotes the ball of radius \( r \) in \( L_p^m \). If \( W = L_p^m \), then \( \gamma_\sigma \triangleq \gamma_{\sigma, L_p^m} \) is called \( L_p \)-gain of the subsystem corresponding to \( \sigma \in \Gamma \).

3. Main results

In this section, we establish conditions under which the \( L_p \) reachability, \( L_p \) observability, and \( L_p \) stability of each subsystem \( \sigma \) guarantee the asymptotic stability of the overall switched system \( \Sigma \). To this end, following some notations used in [7], given a subsystem \( \sigma \in \Gamma \) we define:

\[
V_{\sigma, \alpha} (x) \triangleq \|g_\sigma (\xi_\sigma (\tau, 0, x, 0), 0)\|_p^{p+1} = \left( \int_0^\infty |g_\sigma (\xi_\sigma (\tau, 0, x, 0), 0)|^p \, d\tau \right)^{\frac{1}{p} + 1}
\]

and

\[
V_{\sigma, r} (x) \triangleq \inf_{(t,u) \in U_\sigma (x)} \|u_t\|_p = \inf_{(t,u) \in U_\sigma (x)} \left( \int_0^t |u(\tau)|^p \, d\tau \right)^{\frac{1}{p}}.
\]

where

\[
U_\sigma (x) \triangleq \{(t,u) : u \in L_p^\infty \cap L_p^m, \xi_\sigma (t, 0, 0, u) = x\}
\]

and \( u_t (t) \) denotes the truncation at time \( t \) of the signal \( u \), i.e.,

\[
u_t (t) \begin{cases} u(t) & 0 \leq t < \tau, \\ 0 & \text{otherwise}. \end{cases}
\]
If the subsystem $\sigma$ is locally uniformly $L_p$-reachable and locally uniformly $L_p$-observable, then the functions $V_{\sigma,r}(x)$ and $V_{\sigma,o}(x)$ are well-defined, for $|x| < c_\sigma$. If the reachability and observability properties hold globally, then $V_{\sigma,r}(x)$ and $V_{\sigma,o}(x)$ are well-defined for $x \in \mathbb{R}^n$.

The following lemma, establishing some properties of the functions $V_{\sigma,r}$ and $V_{\sigma,o}$, which we shall exploit to derive Theorem 1, can be proved using similar steps as in the proof of [7, Theorem 3.2].

**Lemma 1.** Let the subsystem $\sigma$ be uniformly locally $L_p$-reachable, uniformly locally $L_p$-observable and finite gain over $B_p(K_\sigma(c_\sigma))$ $L_p$-stable, with gain $\gamma_{\sigma,B_p(K_\sigma(c_\sigma))}$. Then $V_{\sigma,r}(\cdot)$ and $V_{\sigma,o}(\cdot)$ are Lyapunov functions for the system $\sigma$ for $|x| < c_\sigma$, and

\[
\frac{1}{\gamma_{\sigma,B}} \xi_{\sigma}(|x|) \leq V_{\sigma,r}(x) \leq \kappa_{\sigma}(|x|),
\]

\[
v_{\sigma}(|x|) \leq V_{\sigma,o}(x) \leq T_{\sigma,B} \vartheta_{\sigma}(|x|),
\]

where $v_{\sigma} \triangleq \xi_{\sigma}^{p+1}$, $\vartheta_{\sigma} \triangleq \kappa_{\sigma}^{p+1}$, $\gamma_{\sigma,B} \triangleq \gamma_{\sigma,B_p(K_\sigma(c_\sigma))}$, and $T_{\sigma,B} \triangleq \gamma_{p+1}$.

We make the following assumption:

**Assumption 1 (Admissible Family of Subsystems).**

1. The conditions of uniform local $L_p$ reachability and uniform $L_p$ reachability are satisfied uniformly over $\sigma \in \Gamma$.
2. The conditions of uniform local $L_p$ observability and $L_p$ uniform observability are satisfied uniformly over $\sigma \in \Gamma$.
3. The conditions of local $L_p$ stability over $W$ and $L_p$ stability are satisfied uniformly over $\sigma \in \Gamma$.

By Assumption 1(a), there exist a $K$-class function $\kappa$, a constant $\tau \triangleq \sup_{\sigma \in \Gamma} t_\sigma < \infty$, and a constant $c \triangleq \inf_{\sigma \in \Gamma} c_\sigma > 0$ such that if $|x| < c$, then there exists $u \in L_\infty^w$ satisfying:

1. $\|u\|_p \leq \kappa(|x|)$
2. $\xi_{\sigma}(t,0,0,u) = x$

for all $\sigma \in \Gamma$, and analogously for the uniform $L_p$ reachability, with a common $K_\infty$-class function $\kappa$.

By Assumption 1(b), there exist a $K$-class function $\zeta$ and a constant $c \triangleq \inf_{\sigma \in \Gamma} c_\sigma > 0$ such that if $|x| < c$, then $\|g_{\sigma} (\xi_{\sigma}(\cdot,0,x,0,0))\|_p \geq \zeta(|x|)$ for all $\sigma \in \Gamma$, and analogously for the $L_p$ uniform observability, with a common $K_\infty$-class function $\kappa$.

By Assumption 1(c), there exists a constant $\gamma_\infty \triangleq \sup_{\sigma \in \Gamma} \gamma_{\sigma,w} < \infty$ such that $\|y\|_p \leq \gamma_\infty \|u\|_p \forall u \in W$, where $y(t) = g_{\sigma}(\xi_{\sigma}(t,0,0,u(t)),u(t))$ for all $\sigma \in \Gamma$, and analogously for the $L_p$ stability, with a gain $\gamma \triangleq \sup_{\sigma \in \Gamma} \gamma_{\sigma} < \infty$.

Note that the conditions on the suprema are immediately satisfied, for example, when the number of different subsystems is finite.

As we are interested in giving conditions on the subsystems $\sigma \in \Gamma$ and on the family $S$ of switching signals, guaranteeing the asymptotic stability of the switched system $\Sigma$ for every switching signal $s \in S$, we consider the following definition of uniform stability over a given family $S$:

**Definition 4 (Uniform Asymptotic Stability over a Family of Switching Signals).** The equilibrium point $x = 0$ of the system (1) is uniformly asymptotically stable over $S$ if and only if there exist a $KL$-class function $\beta(\cdot,\cdot)$ and a constant $a \in (0,\infty)$, independent of $t_0$, such that, for each $s \in S$,

\[
|x(t)| \leq \beta(|x(t_0)|,t-t_0), \quad \forall t \geq t_0 \geq 0, \forall |x(t_0)| < a.
\]

To guarantee the uniform asymptotic stability over $S$ of the switched system $\Sigma$, besides requiring the $L_p$ reachability, observability and stability of each subsystem, we restrict the family $S$ of switching signals by imposing a lower bound on their dwell times. Given $N_0$, $\tau_D > 0$, we define the following class of functions:

\[
S_{\text{ave}}[\tau_D,N_0] \triangleq \left\{ s \in S : N_s(t'',t') \leq N_0 + \frac{t'' - t'}{\tau_D} \right\},
\]
where \( N_s(t'', t') \) denotes the number of discontinuities of the function \( s \) over the interval \((t', t'')\). The scalars \( \tau_D \) and \( N_0 \) are called average dwell time and chatter bound, respectively. So, \( S_{\text{ave}}[\tau_D, N_0] \) is the family of all switching signals in \( S \) with average dwell time \( \tau_D \) and chatter bound \( N_0 \).

Our main result consists in relating the \( L_p \) reachability, \( L_p \) observability, and \( L_p \) stability properties of the subsystems with the asymptotical stability of the switched system (1) over \( S_{\text{ave}}[\tau_D, N_0] \), for some dwell time and chatter bound dependent on the aforementioned \( L_p \) input–output properties of the subsystems. This is stated in the next theorem, which requires the following definition of admissible functions \( g_\sigma, \sigma \in \Gamma \).

**Definition 5** (Admissible Functions \( g_\sigma \)). The admissible functions \( g_\sigma, \sigma \in \Gamma \) are defined by the following condition:

\[
\text{if } |x_1| \leq |x_2|, \text{ then } |g_\sigma(x_1, 0)| \leq |g_\sigma(x_2, 0)|, \forall \sigma \in \Gamma. \tag{3}
\]

**Theorem 1.** Let the subsystems of the switched system \( \Sigma \) defined by (1) satisfy the Assumption 1 and the condition (3). Then one can find \( t^* > 0 \) such that the switched system \( \Sigma \) is asymptotically stable uniformly over \( S_{\text{ave}}[\tau_D, N_0] \) for every average dwell time \( \tau_D \geq t^* \) and every chatter bound \( N_0 \).

The following two lemmas will be used in the proof of Theorem 1.

**Lemma 2.** Let the subsystems of the switched system \( \Sigma \) defined by (1) satisfy the Assumption 1 and the condition (3). Let \( c > 0 \) and \( B(0, c) \triangleq \{x \in \mathbb{R}^n : |x| < c\} \). Then, for each subsystem (2), there exist a continuously differentiable function \( \tilde{V}_\sigma : B(0, c) \to \mathbb{R} \), a \( K \)-class function \( D_\sigma : B(0, c) \to \mathbb{R} \), a positive constant \( \mu \), and a \( K \)-class functions \( \alpha_1, \tilde{\alpha}_1 \) such that:

\[
\begin{align*}
\frac{\partial \tilde{V}_\sigma}{\partial x} f_\sigma(x, 0) &\leq -D_\sigma(x) \quad \tag{4} \\
\alpha_1(|x|) &\leq \tilde{V}_\sigma(x) \leq \tilde{\alpha}_1(|x|) \quad \tag{5}
\end{align*}
\]

for all \( x \in B(0, c) \) and all \( \sigma, \sigma' \in \Gamma \).

**Proof.** By Assumption 1(c), for each \( \sigma \in \Gamma \) we can choose as a candidate Lyapunov function:

\[
\tilde{V}_\sigma(x) \triangleq V_{\sigma, o}(x) = \|g_\sigma(\xi_\sigma(t, 0, x, 0), 0)\|_p^{p+1} = \left( \int_0^\infty |g_\sigma(\xi_\sigma(t, 0, x, 0), 0)|^p \, d\tau \right)^{\frac{1}{p} + 1}.
\]

As

\[
\begin{align*}
\tilde{V}_\sigma(x) &= \tilde{V}(\xi_\sigma(t, 0, x_0, 0)) \\
&= \left( \int_0^\infty |g_\sigma(\xi_\sigma(t, 0, x_0, 0), 0)|^p \, d\tau \right)^{\frac{1}{p} + 1} \\
&= \left( \int_t^\infty |g_\sigma(\xi_\sigma(t, 0, x_0, 0), 0)|^p \, d\tau \right)^{\frac{1}{p} + 1},
\end{align*}
\]

we get

\[
\dot{\tilde{V}}_\sigma(x) = \frac{\partial V_{\sigma, o}(x)}{\partial x} f_\sigma(x, 0)
\]

\[
= -\left( \frac{1}{p} + 1 \right) \left( \int_0^\infty |g_\sigma(\xi_\sigma(t, 0, x_0, 0), 0)|^p \, d\tau \right)^{\frac{1}{p}} |g_\sigma(\xi_\sigma(t, 0, x_0, 0), 0)|^p
\]

\[
\leq -\left( \frac{1}{p} + 1 \right) \zeta(|x|) |g_\sigma(x, 0)|^p \triangleq -D_\sigma(x).
\]

By (3), \( D_\sigma \) is a \( K \)-class function, thus (4) is proved.
The inequality (5) follows from Assumptions 1(a) and 1(b) and Lemma 1 with \( v \triangleq \xi^{p+1}, \vartheta \triangleq \kappa^{p+1}, \gamma_{\sigma, B, \rho(\sigma, \vartheta)} = \{ |x| = v(|x|), \text{ and } \tilde{\alpha}_1(|x|) = \gamma_B \vartheta(|x|) \}. \]

**Lemma 3.** Let the subsystems of the switched system \( \Sigma \) defined by (1) satisfy the Assumption 1 and the condition (3). Let \( c > 0 \) and \( B(0, c) \triangleq \{ x \in \mathbb{R}^n : |x| < c \} \). Then, for each \( \sigma \in \Gamma \), one can find a \( C^1 \) function \( \rho : B \rightarrow [0, +\infty) \) of class \( K \), where \( B \subset B(0, c) \) is a compact set, such that there exist a \( \alpha \)-class functions \( \alpha_1 \triangleq \rho \circ \alpha_1 \) and \( \alpha_2 \triangleq \rho \circ \tilde{\alpha}_1 \) such that:

(i) \( \frac{\partial V_\sigma}{\partial x} f_\sigma(x, 0) \leq -\lambda_0 V_\sigma(x) \)

(ii) \( \alpha_2(|x|) \leq V_\sigma(x) \leq \alpha_2(|x|) \)

for all \( x \in B \) and \( \sigma, \sigma' \in \Gamma \).

Moreover,

(iii) \( V_\sigma(x) \leq \mu V_{\sigma'}(x) \)

where \( \mu \triangleq \sup_{|x| < c, x \neq 0} \left\{ \frac{\rho \circ \gamma_B \vartheta(|x|)}{\rho \circ v(|x|)} \right\} \).

**Proof.** The inequalities (i) and (ii) follow from Lemma 2 and [27, Lemma 11, p. 22, Lemma 12, p. 23, and Proposition 13, p. 24].

Let us now prove (iii). By Lemma 1 and Assumptions 1(a) and 1(b), for every \( \sigma, \sigma' \in \Gamma \) one has \( V_{\sigma, 0}(x) \leq \gamma_B \vartheta(|x|) \) and \( V_{\sigma', 0}(x) \geq v(|x|) \). Thus:

\[
\sup_{\sigma, \sigma' \in \Gamma, |x| < c, x \neq 0} \left\{ \frac{\rho \circ \tilde{V}_\sigma(x)}{\rho \circ \tilde{V}_{\sigma'}(x)} \right\} \leq \sup_{|x| < c, x \neq 0} \left\{ \frac{\rho \circ \gamma_B \vartheta(|x|)}{\rho \circ v(|x|)} \right\}
\]

(6)

**Proof of Theorem 1.** As the inequalities (i), (ii), and (iii) of Lemma 3 hold, we first follow similar steps as in the proof of [26, Theorem 2].

To this end, let us consider a switching signal \( s(t) \) such that \( s(t) = \sigma_k \) for all \( t \in [t_k, t_{k+1}) \). Recalling that \( T_{s, x_0} \) denotes the maximum interval of definition of the trajectory \( x(t) = \xi_s(t, 0, x_0, 0) \), the following two cases have to be considered: \( T_{s, x_0} \leq t_0 \) and \( T_{s, x_0} > t_0 \) for some \( k \in \mathbb{N} \).

In the first case, only the subsystem \( \sigma_0 \) becomes active, i.e., the trajectory ceases to be defined before the first switching occurs. Then, using the same arguments as in [7], it is easy to show that \( x(t) \) evolves in a fixed compact set of \( \mathbb{R}^n \), for \( t \in [0, T_{s, x_0}) \). Hence, from standard results of differential equations (see, e.g., [28, Appendix C]), \( x(t) \) can be prolonged on \([0, t_0) \). At this point, the switching occurs, and a new maximum interval of definition, \( T_{s, x_0} \), is defined for the trajectory. The situation \( T_{s, x_0} > t_0 \), falls into the second case.

Hence, from now on, let \( T_{s, x_0} > t_0 \) for some \( k \in \mathbb{N} \), and take a time instant \( T \) such that \( 0 \leq t_0 < T \). As \( N_s(T, t_0) \) denotes the number of discontinuities of the signal \( s(t) \) in the interval \((t_0, T)\), we can let \( t_{N_s(T, t_0)+1} \triangleq T \). Since \( s(\cdot) \) is piecewise constant, the function \( v(t) \triangleq e^{\lambda_0 t} V_{\xi_s}(x(t)) \) is piecewise continuously differentiable and

\[
\dot{v} = \lambda_0 v(t) + e^{\lambda_0 t} \frac{\partial V_{\sigma_k}}{\partial x} f_{\sigma_k}(x, 0), \quad t \in [t_k, t_{k+1}), \quad k = 0, 1, \ldots, N_s(T, t_0).
\]

By Lemma 3(i), in the time interval \((t_k, t_{k+1})\) one has \( \dot{v}(t) \leq 0 \), so

\[
v(t) \leq v(t_k), \quad t \in [t_k, t_{k+1}), \quad k = 0, 1, \ldots, N_s(T, t_0).
\]

(7)

Hence, by Lemma 3(iii), for \( k = 0, 1, \ldots, N_s(T, t_0) \) we obtain

\[
v(t_{k+1}) = e^{\lambda_0 t_{k+1}} V_{\sigma_{k+1}}(x(t_{k+1})) \leq e^{\lambda_0 t_{k+1}} V_{\sigma_k}(x(t_{k+1}))
\]

and, using the continuity of \( V_{\sigma_k} \) and \( x \),

\[
v(t_{k+1}) \leq \mu \lim_{t \rightarrow t_{k+1}} e^{\lambda_0 t} V_{\sigma_k} (x(t)) = \mu \lim_{t \rightarrow t_{k+1}} v(t).
\]

As (7) holds, for \( k = 0, 1, \ldots, N_s(T, t_0) \) one has \( \lim_{t \rightarrow t_{k+1}} v(t) \leq v(t_k) \) and, since \( v \) is continuous, \( v(t_{k+1}) \leq \mu v(t_k) \).
By applying this inequality for \( k = 0, 1, \ldots, N_s(T, t_0) - 1 \), one obtains
\[
v(t_{N_s(T, t_0)}) \leq \mu^{N_s(T, t_0)} v(t_0).
\]

By (7) and the continuity of \( v \), one also gets
\[
e^{\lambda_0 T} V_{x(T^-)}(x(T)) = \lim_{t \to T^-} v(t) \leq v(t_{N_s(T, t_0)}) \leq \mu^{N_s(T, t_0)} v(t_0) = \mu^{N_s(T, t_0)} e^{\lambda_0 t_0} V_{0_0}(x(t_0)),
\]
where \( x(T^-) \triangleq \lim_{T \to T^-} x(t) \).

Multiplying the above-written inequality for \( e^{\lambda_0 T} \) gives
\[
V_{x(T^-)}(x(T)) \leq e^{-\lambda_0 (T-t_0) + N_s(T, t_0) \log \mu} V_{0_0}(x(t_0)).
\]

Therefore, using Lemma 3(ii) one concludes that, for every switching signal \( s \in S \) and every time instant \( 0 \leq t_0 < T < T_{s, x_0} \),
\[
|x(T)| \leq \frac{\alpha_2^{-1}}{\alpha_2} \left[ e^{-\lambda_0 (T-t_0) + N_s(T, t_0) \log \mu} \tilde{\alpha}_2(|x(t_0)|) \right] = (\rho \circ v)^{-1} \left[ e^{-\lambda_0 (T-t_0) + N_s(T, t_0) \log \mu} \mathcal{B}_{\rho} \circ \vartheta(|x(t_0)|) \right] = (\rho \circ v)^{-1} \left[ e^{-\delta} \mathcal{B}_{\rho} \circ \vartheta(|x(t_0)|) \right],
\]
where \( \delta \triangleq \lambda_0 (T-t_0) - N_s(T, t_0) \log \mu \) and \( \mu \geq 1 \) is defined in (6).

When \( \mu > 1 \), take \( \tau_D \geq t^* \triangleq \frac{\log \mu}{\lambda_0} \) if \( \mu > 1 \), with \( \lambda \in (0, \lambda_0) \) and \( s \in S_{\text{ave}}[\tau_D, N_0] \). The case \( \mu = 1 \) corresponds to the existence of a common Lyapunov function for all the subsystems; thus, in such a case any switching signal \( s \in S \), with no constraints on the dwell time, provides an asymptotically stable behaviour for the switched system.

From the above-written choice for \( \tau_D \), one has
\[
N_s(T, t_0) \leq N_0 + \frac{\lambda_0 - \lambda}{\log \mu} (T - t_0)
\]
and then it turns out that:
\[
|x(T)| \leq \beta(|x(t_0)|, T - t_0) \quad \text{for} \quad T \in [0, T_{s, x_0}),
\]
where \( \beta(\chi, t) \triangleq (\rho \circ v)^{-1} \left[ e^{|N_0 \log \mu - \lambda t|} \mathcal{B}_{\rho} \circ \vartheta(\chi) \right] \) is of class \( KL \).

Before proceeding, note that, to guarantee \( x(T) \) to be in the domain of validity of the assumptions, it has to be \( |x(T)| < c \), i.e.,
\[
(\rho \circ v)^{-1} \left[ e^{-\delta} \mathcal{B}_{\rho} \circ \vartheta(|x(t_0)|) \right] < c.
\]

As \( N_s(T, t_0) \leq N_0 + \frac{T-t_0}{\lambda_0} \) and \( \tau_D \geq \frac{\log \mu}{\lambda_0} \), one obtains \( \delta \geq -N_s(T, t_0) \log \mu \geq -N_0 \log \mu + \lambda(T - t_0) \geq 0 \). Then, it is sufficient that
\[
(\rho \circ v)^{-1} \left[ e^{N_0 \log \mu} \mathcal{B}_{\rho} \circ \vartheta(|x(t_0)|) \right] < c,
\]
i.e.
\[
|x(t_0)| < \min \left\{ c, (\rho \circ \vartheta)^{-1} \left[ e^{-N_0 \log \mu} \mathcal{B}_{\rho} \circ v(e) \right] \right\}.
\]

As \( x(t) = \xi_s(t_0, x_0, 0) \) (and, in particular, \( \xi_{0_0}(t, t_k, x(t_k), 0) \)) evolves in a fixed compact subset of \( \Re^n \) for \( t \in [0, T) \), by standard results of differential equations [28, Appendix C], one can prolong the trajectory on \([0, T_{k+1}]\) (recall that \( T_{s, x_0} \) has been defined such that \( T_{s, x_0} > t_k \)). When a new switching occurs, the maximal interval of definition becomes \( T_{s, x_0} > t_{k+1} \). Iterating the arguments above, it follows that the trajectory is defined for all \( t \geq 0 \). Therefore, we obtain
\[
|x(t)| \leq \beta(|x(t_0)|, t - t_0) \quad \text{for} \quad t \geq 0,
\]
which concludes the proof of Theorem 1. |

\[\square\]
4. Conclusions

In this work, we have considered continuous-time nonlinear switched systems. The switching occurs among a class of subsystems that are defined by certain input–output properties, stated in terms of $L_p$ spaces of signals. We have investigated the relationships between the $L_p$ stability of each subsystem and the internal stability of the switched system. As a result of our analysis, we have found conditions on the dwell time of the switching signals, related to the $L_p$ reachability, observability, and stability of the subsystems, that guarantee the local asymptotic stability of the overall system.

With respect to available results on this subject, our work provides the following novelties. First, the notion of $L_p$ input–output stability is applied to a quite general class of nonlinear switched systems, whereas, usually, only classical Lyapunov stability is considered. Second, notions of $L_p$ reachability and observability are considered in order to find a lower bound on the average dwell time, which defines a class of switched systems. Third, for such systems the local asymptotic stability is proved to hold under conditions expressed in terms of from the $L_p$ input–output stability, reachability, and observability properties of the subsystems.

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